Tutorial IX

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Happy New Year!

The goal for today is to get as much logic as possible under your belt (with a view towards Assignment 9:

What is a logic?

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A logic is a formal system consisting of a formal language with syntax and semantics (like German) and a proof calculus (unlike German). It is formal in the sense that it is possible to decide unambiguously (in principle) whether a formula is well-formed, whether a proposition is satisfiable/valid and whether it is derivable (again unlike German).

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More formally:

Definition (Formal System)

A formal system is a triple $S := \langle \mathcal{L}, \mathcal{K}, \models, \vdash \rangle$ where $\mathcal{L}$ is a formal language, $\mathcal{K}$ is a set, $\models \subseteq \mathcal{K} \times \mathcal{L}$ and $\vdash$ is

- proof-reflexive ($\mathcal{H} \vdash A$ if $A \in \mathcal{H}$);
- proof-transitive ($\mathcal{H} \vdash A$ and $\mathcal{H}' \cup A \vdash B$ then $\mathcal{H} \cup \mathcal{H}' \vdash B$);
- monotonic (it admits weakening, i.e. if $\mathcal{H} \vdash A$ and $\mathcal{H} \subseteq \mathcal{H}'$ then $\mathcal{H}' \vdash A$).

Example

Example (A trivial logic)

- $\mathcal{L} = \{A\}$
- $K = \{\langle \mathcal{D}_o, \mathcal{I} \rangle : \mathcal{D}_o = \{T, F\}, \mathcal{I} : \mathcal{L} \longrightarrow \mathcal{D}_o\}$
- $X \vDash Y$ iff for all $\mathcal{M} \in \mathcal{K}$, $\mathcal{I}(A) = T$ implies $\mathcal{I}(B) = T$
- $\vdash A$

Is A satisfiable? Valid? Falsifiable? Is $\vdash$ proof-reflexive? Proof-transitive? Monotonic? Sound? Complete? (See slides for definitions).

Natural Deduction Rules

$$\frac{A \wedge B}{A}\,\wedge E_l \qquad \frac{A \wedge B}{B}\,\wedge E_r \qquad \frac{A \quad B}{A \wedge B}\,\wedge I \qquad \frac{A \Rightarrow B \quad A}{B}\,{\Rightarrow}E \qquad \frac{\begin{array}{c}[A]^1\\ \vdots\\ B\end{array}}{A \Rightarrow B}\,{\Rightarrow}I^1$$

$$\frac{A \vee B \quad \begin{array}{c}[A]^1\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \vdots\\ C\end{array}}{C}\,\vee I^{12} \qquad \frac{A}{A \vee B}\,\vee I_l \qquad \frac{B}{A \vee B}\,\vee I_r$$

$$\frac{\begin{array}{c}[A]^1\\ \vdots\\ \bot\end{array}}{\neg A}\,\neg I^1 \qquad \frac{\neg\neg A}{A}\,\neg E \qquad \frac{\neg A \quad A}{\bot}\,\bot I \qquad \frac{\bot}{A}\,\bot E$$

Natural Deduction II

(Note: we also need the Tertium non Datur axiom $A \vee \neg A$)
Prove using the Natural Deduction Calculus above:

- $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
- $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \wedge B) \Rightarrow C$

Deduction Theorem & Unsatisfiability Theorem

Prove the following Theorems:

Theorem (Deduction Theorem)

$\mathcal{H}, A \vdash B$ *iff* $\mathcal{H} \vdash A \Rightarrow B$

Theorem (Unsatisfiability (Contradiction) Theorem)

$\mathcal{H} \vDash A$ *iff* $\mathcal{H} \cup \{\neg A\}$ *is unsatisfiable*