Künstliche Intelligenz I
Homework 11 - Unofficial Solution

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NO GARUANTEE FOR CORRECTNESS!

1 Problem 11.1

Let ϕ_1, ϕ_2 be two variable assignments and **A** a first-order formula. Prove: If $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(A)$, then $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$.

Induction over the depth of **A**:

At first a small Lemma for terms:
Let **T** be a term and n be the depth of **T** (the number of functions in **T**)

Base Case $n = 0$:
T is one variable and because there is no quantifier, it is free. So $\phi_1(T) = \phi_2(T)$ is valid as well as $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

Induction hypothesis n:
Let ϕ_1, ϕ_2 be two variable assignments and **T** a first-order term with depth n. If $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(A)$, then $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

Inductive step $n \to n+1$:
$\mathbf{T} = f(\mathbf{T}_1, \ldots, \mathbf{T}_n)$ where $\mathbf{T}_1, \ldots \mathbf{T}_n$ have all the depth n and since there are no quantifiers, all variable occurrences are free. So if $\phi_1(X) = \phi_2(X)$ for all free variables X in $\mathbf{T}, \mathbf{T}_1, \ldots \mathbf{T}_n$ then obviously $\phi_1(X) = \phi_2(X)$ for all variables X in $\mathbf{T}, \mathbf{T}_1, \ldots \mathbf{T}_n$
So $\mathcal{I}_{\phi_1}(\mathbf{T}) = \mathcal{I}(f)(\mathcal{I}_{\phi_1}(\mathbf{T}_1), \ldots, (\mathcal{I}_{\phi_1}(\mathbf{T}_n)) \overset{Ih}{=} \mathcal{I}(f)(\mathcal{I}_{\phi_2}(\mathbf{T}_1), \ldots, (\mathcal{I}_{\phi_2}(\mathbf{T}_n)) = \mathcal{I}_{\phi_2}(\mathbf{T})$ □

Now the real proof:

Let **A** be a first order logic formula and n be the depth of **A** (the number of connectors in **A**)

Base Cases $n = 0$:
A is $\top$ (or $\bot$). Here it is trivial that $\phi_1(A) = \phi_2(A)$ is valid as well as $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

A is an atomic proposition and because there is no quantifier in it, all variable occurrences are free. So if $\phi_1(X) = \phi_2(X)$ for all free variables X in **A** then obviously $\phi_1(X) = \phi_2(X)$ for all variables X in **A**. This concludes to $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$. Concerning the terms which construct the proposition, see the lemma above.

Induction hypothesis n:
Let ϕ_1, ϕ_2 be two variable assignments and **A** a first-order formula with depth n. If $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(A)$, then $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$

Inductive steps $n \to n+1$:
1. $\mathbf{A} = \mathbf{B} \wedge \mathbf{C}$ where **B,C** are first order formulas with the depth n
Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{A})$.
Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{B})$ and $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{C})$ because $free(\mathbf{C}) \subset free(\mathbf{A})$ and $free(\mathbf{B}) \subset free(\mathbf{A})$.

So $\mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}(\wedge)(\mathcal{I}_{\phi_1}(\mathbf{B}), (\mathcal{I}_{\phi_1}(\mathbf{C})) \stackrel{Ih}{=} \mathcal{I}(\wedge)(\mathcal{I}_{\phi_2}(\mathbf{B}), (\mathcal{I}_{\phi_2}(\mathbf{C})) = \mathcal{I}_{\phi_2}(\mathbf{A})$ □

2. $\mathbf{A} = \neg\mathbf{B}$ where **B** is a first order formula with the depth n
Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{A})$.
Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{B})$ because $free(\mathbf{B}) = free(\mathbf{A})$.

So $\mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}(\neg)(\mathcal{I}_{\phi_1}(\mathbf{B})) \stackrel{Ih}{=} \mathcal{I}(\neg)(\mathcal{I}_{\phi_2}(\mathbf{B})) = \mathcal{I}_{\phi_2}(\mathbf{A})$ □

3. $\mathbf{A} = \forall Y.\mathbf{B}$ where **B** is a first order formula with the depth n
Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{A})$.
Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in free(\mathbf{B})\backslash Y$ because $free(\mathbf{B})\backslash Y = free(\mathbf{A})$.

So $\mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}_{\phi_1}(\forall Y.\mathbf{B}) = [\top \text{ iff } \mathcal{I}_{\phi_1,[a/Y]}(B) = \top \text{ for all } a \in \mathcal{D}_\iota]$

At this point all occurrences of Y get substituted so that $free(B) = free(B)\backslash Y$, so that with the definition of ϕ_1, ϕ_2 the **induction hypothesis** is applicable.

$\stackrel{Ih}{=} [\top \text{ iff } \mathcal{I}_{\phi_2,[a/Y]}(B) = \top \text{ for all } a \in \mathcal{D}_\iota] = \mathcal{I}_{\phi_2}(\forall Y.\mathbf{B}) = \mathcal{I}_{\phi_2}(\mathbf{A})$ □

□

2 Problem 11.2

I assume here that $\mathcal{I}(\vee), \mathcal{I}(\exists)$ are easily constructed from the other Interpretations.
1.

$$\forall X.\forall Y.X + Y = X + Y$$
is the same as $\forall X.\forall Y = (X+Y, X+Y)$
let $\mathcal{I}_\phi$ be any Interpretation :
$$\mathcal{I}_\phi(\forall X.\forall Y = (X+Y, X+Y)) \stackrel{?}{=} \top$$
$$\Leftrightarrow \top \text{ iff } \mathcal{I}_{\phi,[a/X]}(\forall Y = (X+Y, X+Y)) = \top \text{ for all } a \in \mathcal{D}_\iota \stackrel{?}{=} \top$$
So the question is $\mathcal{I}_{\phi,[a/X]}(\forall Y = (X+Y, X+Y)) \stackrel{?}{=} \top$ for all $a \in \mathcal{D}_\iota$
That translates into ($\top$ iff $\mathcal{I}_{\phi,[b/Y]}(= (a+Y, a+Y)) = \top$ for all $a \in \mathcal{D}_\iota) \stackrel{?}{=} \top$
So the question is $\mathcal{I}_\phi(= (a+b, a+b)) \stackrel{?}{=} \top$ for all $a, b \in \mathcal{D}_\iota$
$\Leftrightarrow \top$, iff $\langle \mathcal{I}_\phi(a+b), \mathcal{I}_\phi(a+b)\rangle \in \mathcal{I}(=)$ for all $a, b \in \mathcal{D}_\iota \stackrel{?}{=} \top$
So the question is: $\langle \mathcal{I}_\phi(a+b), \mathcal{I}_\phi(a+b)\rangle \stackrel{?}{\in} \mathcal{I}(=)$ for all $a, b \in \mathcal{D}_\iota$
Since $\mathcal{I}(=)$ is not in the semantic of the first order logic without equality, I can assume that:
$\mathcal{I}(=) = \{(x, y) | x, y \in \mathcal{D}_\iota \wedge \text{x and y are not the same element}\}$
So the answer is $\langle \mathcal{I}_\phi(a+b), \mathcal{I}_\phi(a+b)\rangle \notin \mathcal{I}(=)$ for all $a, b \in \mathcal{D}_\iota$
Thus $\mathcal{I}_\phi(\forall X.\forall Y = (X+Y, X+Y)) \neq \top$
Which means that this formula is not valid!A model in which it is false:
$\mathcal{M} = \langle \mathcal{D}_\iota, \mathcal{I}\rangle$ where $\mathcal{D}_\iota = \mathbb{N}$ and $\mathcal{I} =$ any Interpretation in which $\mathcal{I}(=)$ is defined as above

2.

$$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$$
$$\exists X.(\neg P(X) \vee \forall Y.P(Y))$$
$$\exists X.\neg(P(X) \wedge \neg\forall Y.P(Y))$$

let $\mathcal{I}_\phi$ be any Interpretation :

$$\mathcal{I}_\phi(\exists X.\neg(P(X) \wedge \neg\forall Y.P(Y))) \stackrel{?}{=} \top$$
$$\stackrel{\mathcal{I}(\exists)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } \mathcal{I}_\phi(\neg(P(a) \wedge \neg\forall Y.P(Y))) \stackrel{?}{=} \top$$
$$\stackrel{\mathcal{I}(\neg)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } \mathcal{I}_\phi((P(a) \wedge \neg\forall Y.P(Y))) \stackrel{?}{=} \bot$$
$$\stackrel{\mathcal{I}(\wedge)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } [\mathcal{I}_\phi(P(a)) \stackrel{?}{=} \bot \text{ or } \mathcal{I}_\phi(\neg\forall Y.P(Y))) \stackrel{?}{=} \bot]$$
$$\stackrel{\mathcal{I}(P)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } [\mathcal{I}_\phi(a) \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(\neg\forall Y.P(Y))) \stackrel{?}{=} \bot]$$
$$\stackrel{\mathcal{I}(\neg),\mathcal{I}(a)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } [\, a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(\forall Y.P(Y))) \stackrel{?}{=} \top]$$
$$\stackrel{\mathcal{I}(\forall)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } [\, a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(P(b)) \stackrel{?}{=} \top \text{ for all } b \in \mathcal{D}_\iota]$$
$$\stackrel{\mathcal{I}(P),\mathcal{I}(b)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_\iota \text{ so that } [\, a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } b \stackrel{?}{\in} \mathcal{I}(P) \text{ for all } b \in \mathcal{D}_\iota]$$

This is true, since either all elements of $\mathcal{D}_\iota$ are in $\mathcal{I}(P)$
which makes the second half of the disjunction true for any chosen a
or there is at least one element which is not in $\mathcal{I}(P)$ and if we choose this element for a
the first half of the disjunction is true.
So it was shown that this formula is valid.

3.

$$P(Y) \Rightarrow \exists X.P(X))$$
$$\neg P(Y) \vee \exists X.P(X)$$

let $\mathcal{I}_\phi$ be any Interpretation :

$$\mathcal{I}_\phi(\neg P(Y) \vee \exists X.P(X)) \stackrel{?}{=} \top$$
$$\stackrel{\mathcal{I}(\vee)}{\Leftrightarrow} [\mathcal{I}_\phi(\neg P(Y)) \stackrel{?}{=} \top \text{ or } \mathcal{I}_\phi(\exists X.P(X)) \stackrel{?}{=} \top]$$
$$\stackrel{\mathcal{I}(\neg),\mathcal{I}(\exists)}{\Leftrightarrow} [\mathcal{I}_\phi(P(Y)) \stackrel{?}{=} \bot \text{ or there is one } a \in \mathcal{D}_\iota \text{ so that } \mathcal{I}_\phi(P(a)) \stackrel{?}{=} \top]$$
$$\stackrel{\mathcal{I}(P),\mathcal{I}(a)}{\Leftrightarrow} [\, \phi(Y) \stackrel{?}{\notin} \mathcal{I}(P) \text{ or there is one } a \in \mathcal{D}_\iota \text{ so that } a \stackrel{?}{\in} \mathcal{I}(P)]$$

This is obviously true (either $\phi(Y) \notin \mathcal{I}(P)$ or
$\phi(Y) \in \mathcal{I}(P)$ and then there choose $a = \phi(Y)$), so the formula is valid.

3 Problem 11.3

1.

$$\forall I\;\dfrac{\Rightarrow I^1\;\dfrac{\exists I\;\dfrac{[Z \le -Z]^1}{\exists Y.Z \le Y}}{Z \le -Z \Rightarrow \exists Y.Z \le Y}}{\forall X.X \le -X \Rightarrow \exists Y.X \le Y}$$

2. Lemma:

$$\neg E\;\dfrac{\neg I\;\dfrac{\bot I\;\dfrac{\forall I\;\dfrac{\neg E\;\dfrac{\neg I\;\dfrac{\bot I\;\dfrac{\exists I\;\dfrac{\text{Ax}\;\dfrac{}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y), \neg P(Z) \vdash \neg P(Z)}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y), \neg P(Z) \vdash \exists Y.\neg P(Y)} \quad \text{Ax}\;\dfrac{}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y), \neg P(Z) \vdash \neg\exists Y.\neg P(Y)}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y), \neg P(Z) \vdash \bot}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y) \vdash \neg\neg P(Z)}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y) \vdash P(Z)}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y) \vdash \forall Y.P(Y)} \quad \text{Ax}\;\dfrac{}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y) \vdash \neg\forall Y.P(Y)}}{\neg\forall Y.P(Y), \neg\exists Y.\neg P(Y) \vdash \bot}}{\neg\forall Y.P(Y) \vdash \neg\neg\exists Y.\neg P(Y)}}{\neg\forall Y.P(Y) \vdash \exists Y.\neg P(Y)}$$

main proof:

$$\vee E\;\dfrac{\exists E\;\dfrac{\exists I\;\dfrac{\Rightarrow I\;\dfrac{\bot E\;\dfrac{\bot I\;\dfrac{\text{Ax}\;\dfrac{}{\neg P(c), P(c) \vdash P(c)} \quad \text{Ax}\;\dfrac{}{\neg P(c), P(c) \vdash \neg P(c)}}{\neg P(c), P(c) \vdash \bot}}{\neg P(c), P(c) \vdash \forall Y.P(Y)}}{\neg P(c) \vdash P(c) \Rightarrow \forall Y.P(Y)}}{\neg P(c) \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))} \quad \text{lemma}\;\dfrac{}{\neg\forall Y.P(Y) \vdash \exists Y.\neg P(Y)}}{\neg\forall Y.P(Y) \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))} \quad \exists I\;\dfrac{\Rightarrow I\;\dfrac{\text{Ax}\;\dfrac{}{\forall Y.P(Y), P(Z) \vdash \forall Y.P(Y)}}{\forall Y.P(Y) \vdash P(Z) \Rightarrow \forall Y.P(Y)}}{\forall Y.P(Y) \vdash \exists X.P(X) \Rightarrow \forall Y.P(Y)} \quad \text{TND}\;\dfrac{}{\emptyset \vdash \forall Y.P(Y) \vee \neg\forall Y.P(Y)}}{\emptyset \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))}$$