

Künstliche Intelligenz I

Homework 11 - Unofficial Solution

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NO GARUANTEE FOR CORRECTNESS!

1 Problem 11.1

Let ϕ_1, ϕ_2 be two variable assignments and \mathbf{A} a first-order formula. Prove: If $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(A)$, then $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$.

Induction over the depth of \mathbf{A} :

At first a small Lemma for terms:

Let \mathbf{T} be a term and n be the depth of \mathbf{T} (the number of functions in \mathbf{T})

Base Case $n = 0$:

\mathbf{T} is one variable and because there is no quantifier, it is free. So $\phi_1(T) = \phi_2(T)$ is valid as well as $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

Induction hypothesis n :

Let ϕ_1, ϕ_2 be two variable assignments and \mathbf{T} a first-order term with depth n . If $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(A)$, then $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

Inductive step $n \rightarrow n + 1$:

$\mathbf{T} = f(\mathbf{T}_1, \dots, \mathbf{T}_n)$ where $\mathbf{T}_1, \dots, \mathbf{T}_n$ have all the depth n and since there are no quantifiers, all variable occurrences are free. So if $\phi_1(X) = \phi_2(X)$ for all free variables X in $\mathbf{T}, \mathbf{T}_1, \dots, \mathbf{T}_n$ then obviously $\phi_1(X) = \phi_2(X)$ for all variables X in $\mathbf{T}, \mathbf{T}_1, \dots, \mathbf{T}_n$

So $\mathcal{I}_{\phi_1}(\mathbf{T}) = \mathcal{I}(f)(\mathcal{I}_{\phi_1}(\mathbf{T}_1), \dots, \mathcal{I}_{\phi_1}(\mathbf{T}_n)) \stackrel{Ih}{=} \mathcal{I}(f)(\mathcal{I}_{\phi_2}(\mathbf{T}_1), \dots, \mathcal{I}_{\phi_2}(\mathbf{T}_n)) = \mathcal{I}_{\phi_2}(\mathbf{T}) \quad \square$

Now the real proof:

Let \mathbf{A} be a first order logic formula and n be the depth of \mathbf{A} (the number of connectors in \mathbf{A})

Base Cases $n = 0$:

\mathbf{A} is \top (or \perp). Here it is trivial that $\phi_1(A) = \phi_2(A)$ is valid as well as $\mathcal{I}_{\phi_1}(T) = \mathcal{I}_{\phi_2}(T)$

\mathbf{A} is an atomic proposition and because there is no quantifier in it, all variable occurrences are free. So if $\phi_1(X) = \phi_2(X)$ for all free variables X in \mathbf{A} then obviously $\phi_1(X) = \phi_2(X)$ for all variables X in \mathbf{A} . This concludes to $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$. **This case trivially includes the case where $\mathbf{A} = P(X)$ where \mathbf{P} is a predicate.** Concerning the terms which construct the proposition, see the lemma above.

Induction hypothesis n :

Let ϕ_1, ϕ_2 be two variable assignments and \mathbf{A} a first-order formula with depth n . If $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(A)$, then $\mathcal{I}_{\phi_1}(A) = \mathcal{I}_{\phi_2}(A)$

Inductive steps $n \rightarrow n + 1$:

1. $\mathbf{A} = \mathbf{B} \wedge \mathbf{C}$ where \mathbf{B}, \mathbf{C} are first order formulas with the depth n

Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{A})$.

Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{B})$ and $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{C})$ because $\text{free}(\mathbf{C}) \subset \text{free}(\mathbf{A})$ and $\text{free}(\mathbf{B}) \subset \text{free}(\mathbf{A})$.

$$\text{So } \mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}(\wedge)(\mathcal{I}_{\phi_1}(\mathbf{B}), (\mathcal{I}_{\phi_1}(\mathbf{C}))) \stackrel{Ih}{=} \mathcal{I}(\wedge)(\mathcal{I}_{\phi_2}(\mathbf{B}), (\mathcal{I}_{\phi_2}(\mathbf{C}))) = \mathcal{I}_{\phi_2}(\mathbf{A}) \quad \square$$

2. $\mathbf{A} = \neg \mathbf{B}$ where \mathbf{B} is a first order formula with the depth n

Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{A})$.

Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{B})$ because $\text{free}(\mathbf{B}) = \text{free}(\mathbf{A})$.

$$\text{So } \mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}(\neg)(\mathcal{I}_{\phi_1}(\mathbf{B})) \stackrel{Ih}{=} \mathcal{I}(\neg)(\mathcal{I}_{\phi_2}(\mathbf{B})) = \mathcal{I}_{\phi_2}(\mathbf{A}) \quad \square$$

3. $\mathbf{A} = \forall Y. \mathbf{B}$ where \mathbf{B} is a first order formula with the depth n

Let ϕ_1, ϕ_2 defined in a way, so that $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{A})$.

Trivially $\phi_1(X) = \phi_2(X)$ for all variables $X \in \text{free}(\mathbf{B}) \setminus Y$ because $\text{free}(\mathbf{B}) \setminus Y = \text{free}(\mathbf{A})$.

$$\text{So } \mathcal{I}_{\phi_1}(\mathbf{A}) = \mathcal{I}_{\phi_1}(\forall Y. \mathbf{B}) = [\top \text{ iff } \mathcal{I}_{\phi_1, [a/Y]}(\mathbf{B}) = \top \text{ for all } a \in \mathcal{D}_\iota]$$

At this point all occurrences of Y get substituted so that $\text{free}(\mathbf{B}) = \text{free}(\mathbf{B}) \setminus Y$, so that with the definition of ϕ_1, ϕ_2 the **induction hypothesis** is applicable.

$$\stackrel{Ih}{=} [\top \text{ iff } \mathcal{I}_{\phi_2, [a/Y]}(\mathbf{B}) = \top \text{ for all } a \in \mathcal{D}_\iota] = \mathcal{I}_{\phi_2}(\forall Y. \mathbf{B}) = \mathcal{I}_{\phi_2}(\mathbf{A}) \quad \square$$

\square

2 Problem 11.2

I assume here that $\mathcal{I}(\forall), \mathcal{I}(\exists)$ are easily constructed from the other Interpretations.

1.

$$\forall X. \forall Y. X + Y = X + Y$$

is the same as $\forall X. \forall Y = (X + Y, X + Y)$

let \mathcal{I}_ϕ be any Interpretation :

$$\mathcal{I}_\phi(\forall X. \forall Y = (X + Y, X + Y)) \stackrel{?}{=} \top$$

$$\Leftrightarrow \top \text{ iff } \mathcal{I}_{\phi, [a/X]}(\forall Y = (X + Y, X + Y)) = \top \text{ for all } a \in \mathcal{D}_\iota \stackrel{?}{=} \top$$

$$\text{So the question is } \mathcal{I}_{\phi, [a/X]}(\forall Y = (X + Y, X + Y)) \stackrel{?}{=} \top \text{ for all } a \in \mathcal{D}_\iota$$

$$\text{That translates into } (\top \text{ iff } \mathcal{I}_{\phi, [b/Y]}(= (a + Y, a + Y)) = \top \text{ for all } a \in \mathcal{D}_\iota) \stackrel{?}{=} \top$$

$$\text{So the question is } \mathcal{I}_\phi(= (a + b, a + b)) \stackrel{?}{=} \top \text{ for all } a, b \in \mathcal{D}_\iota$$

$$\Leftrightarrow \top, \text{ iff } \langle \mathcal{I}_\phi(a + b), \mathcal{I}_\phi(a + b) \rangle \in \mathcal{I}(=) \text{ for all } a, b \in \mathcal{D}_\iota \stackrel{?}{=} \top$$

$$\text{So the question is: } \langle \mathcal{I}_\phi(a + b), \mathcal{I}_\phi(a + b) \rangle \stackrel{?}{\in} \mathcal{I}(=) \text{ for all } a, b \in \mathcal{D}_\iota$$

Since $\mathcal{I}(=)$ is not in the semantic of the first order logic without equality, I can assume that:

$$\mathcal{I}(=) = \{(x, y) | x, y \in \mathcal{D}_\iota \wedge x \text{ and } y \text{ are not the same element}\}$$

$$\text{So the answer is } \langle \mathcal{I}_\phi(a + b), \mathcal{I}_\phi(a + b) \rangle \notin \mathcal{I}(=) \text{ for all } a, b \in \mathcal{D}_\iota$$

$$\text{Thus } \mathcal{I}_\phi(\forall X. \forall Y = (X + Y, X + Y)) \neq \top$$

Which means that this formula is not valid! A model in which it is false:

$\mathcal{M} = \langle \mathcal{D}_\iota, \mathcal{I} \rangle$ where $\mathcal{D}_\iota = \mathbb{N}$ and $\mathcal{I} =$ any Interpretation in which $\mathcal{I}(=)$ is defined as above

2.

$$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$$

$$\exists X.(\neg P(X) \vee \forall Y.P(Y))$$

$$\exists X.\neg(P(X) \wedge \neg \forall Y.P(Y))$$

let \mathcal{I}_ϕ be any Interpretation :

$$\mathcal{I}_\phi(\exists X.\neg(P(X) \wedge \neg \forall Y.P(Y))) \stackrel{?}{=} \top$$

$$\stackrel{\mathcal{I}(\exists)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } \mathcal{I}_\phi(\neg(P(a) \wedge \neg \forall Y.P(Y))) \stackrel{?}{=} \top$$

$$\stackrel{\mathcal{I}(\neg)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } \mathcal{I}_\phi((P(a) \wedge \neg \forall Y.P(Y))) \stackrel{?}{=} \perp$$

$$\stackrel{\mathcal{I}(\wedge)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } [\mathcal{I}_\phi(P(a)) \stackrel{?}{=} \perp \text{ or } \mathcal{I}_\phi(\neg \forall Y.P(Y)) \stackrel{?}{=} \perp]$$

$$\stackrel{\mathcal{I}(P)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } [\mathcal{I}_\phi(a) \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(\neg \forall Y.P(Y)) \stackrel{?}{=} \perp]$$

$$\stackrel{\mathcal{I}(\neg), \mathcal{I}(a)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } [a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(\forall Y.P(Y)) \stackrel{?}{=} \top]$$

$$\stackrel{\mathcal{I}(\forall)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } [a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } \mathcal{I}_\phi(P(b)) \stackrel{?}{=} \top \text{ for all } b \in \mathcal{D}_l]$$

$$\stackrel{\mathcal{I}(P), \mathcal{I}(b)}{\Leftrightarrow} \text{is there one } a \in \mathcal{D}_l \text{ so that } [a \stackrel{?}{\notin} \mathcal{I}(P) \text{ or } b \stackrel{?}{\in} \mathcal{I}(P) \text{ for all } b \in \mathcal{D}_l]$$

This is true, since either all elements of \mathcal{D}_l are in $\mathcal{I}(P)$

which makes the second half of the disjunction true for any chosen a

or there is at least one element which is not in $\mathcal{I}(P)$ and if we choose this element for a

the first half of the disjunction is true.

So it was shown that this formula is valid.

3.

$$P(Y) \Rightarrow \exists X.P(X)$$

$$\neg P(Y) \vee \exists X.P(X)$$

let \mathcal{I}_ϕ be any Interpretation :

$$\mathcal{I}_\phi(\neg P(Y) \vee \exists X.P(X)) \stackrel{?}{=} \top$$

$$\stackrel{\mathcal{I}(\vee)}{\Leftrightarrow} [\mathcal{I}_\phi(\neg P(Y)) \stackrel{?}{=} \top \text{ or } \mathcal{I}_\phi(\exists X.P(X)) \stackrel{?}{=} \top]$$

$$\stackrel{\mathcal{I}(\neg), \mathcal{I}(\exists)}{\Leftrightarrow} [\mathcal{I}_\phi(P(Y)) \stackrel{?}{=} \perp \text{ or there is one } a \in \mathcal{D}_l \text{ so that } \mathcal{I}_\phi(P(a)) \stackrel{?}{=} \top]$$

$$\stackrel{\mathcal{I}(P), \mathcal{I}(a)}{\Leftrightarrow} [\phi(Y) \stackrel{?}{\notin} \mathcal{I}(P) \text{ or there is one } a \in \mathcal{D}_l \text{ so that } a \stackrel{?}{\in} \mathcal{I}(P)]$$

This is obviously true (either $\phi(Y) \notin \mathcal{I}(P)$ or

$\phi(Y) \in \mathcal{I}(P)$ and then there choose $a = \phi(Y)$), so the formula is valid.

3 Problem 11.3

1.

$$\Rightarrow I^1 \frac{\exists I \frac{[X \leq -X]^1}{\exists Y.X \leq Y}}{X \leq -X \Rightarrow \exists Y.X \leq Y} \\ \forall I \frac{\quad}{\forall X.(X \leq -X \Rightarrow \exists Y.X \leq Y)}$$

2. Lemma:

$$\frac{\text{Ax} \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y), \neg P(Z) \vdash \neg P(Z)} \quad \text{Ax} \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y), \neg P(Z) \vdash \neg \exists Y.\neg P(Y)}}{\exists I \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y), \neg P(Z) \vdash \exists Y.\neg P(Y)}}}{\perp I \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y), \neg P(Z) \vdash \perp}} \\ \frac{\neg I \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y) \vdash \neg \neg P(Z)} \quad \neg E \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y) \vdash P(Z)}}{\forall I \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y) \vdash \forall Y.P(Y)}}}{\perp I \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y) \vdash \perp}} \quad \text{Ax} \frac{\quad}{\neg \forall Y.P(Y), \neg \exists Y.\neg P(Y) \vdash \neg \forall Y.P(Y)} \\ \frac{\neg I \frac{\quad}{\neg \forall Y.P(Y) \vdash \neg \neg \exists Y.\neg P(Y)}}{\neg E \frac{\quad}{\neg \forall Y.P(Y) \vdash \exists Y.\neg P(Y)}}$$

main proof:

$$\frac{\text{Ax} \frac{\quad}{\neg P(c), P(c) \vdash P(c)} \quad \text{Ax} \frac{\quad}{\neg P(c), P(c) \vdash \neg P(c)}}{\perp I \frac{\quad}{\neg P(c), P(c) \vdash \perp}} \\ \frac{\perp E \frac{\quad}{\neg P(c), P(c) \vdash \forall Y.P(Y)}}{\Rightarrow I \frac{\quad}{\neg P(c) \vdash P(c) \Rightarrow \forall Y.P(Y)}} \quad \text{lemma} \frac{\quad}{\neg \forall Y.P(Y) \vdash \exists Y.\neg P(Y)} \quad \Rightarrow I \frac{\text{Ax} \frac{\quad}{\forall Y.P(Y), P(Z) \vdash \forall Y.P(Y)}}{\forall Y.P(Y) \vdash P(Z) \Rightarrow \forall Y.P(Y)} \\ \frac{\exists I \frac{\quad}{\neg P(c) \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))} \quad \text{lemma} \frac{\quad}{\neg \forall Y.P(Y) \vdash \exists Y.\neg P(Y)}}{\exists E \frac{\quad}{\neg \forall Y.P(Y) \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))}} \quad \exists I \frac{\quad}{\forall Y.P(Y) \vdash \exists X.P(X) \Rightarrow \forall Y.P(Y)} \quad \text{TND} \frac{\quad}{\emptyset \vdash \forall Y.P(Y) \vee \neg \forall Y.P(Y)} \\ \frac{\quad}{\forall E \frac{\quad}{\neg \forall Y.P(Y) \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))}} \quad \frac{\quad}{\emptyset \vdash \exists X.(P(X) \Rightarrow \forall Y.P(Y))}$$